The Ising and Spin O(n) models

Thursday, 17 June 2021 13:10

Classical statistical mechanics - Magnetism

Spontaneous magnetization at low temperatures

Reference: Lectures on the spin and loop O(n) models / Peled-Spinka

Additional reference: Statistical mechanics of lattice systems: a concrete mathematical introduction / Friedli-Velenik

Models

Given a finite graph $G=(V(G),E(G))$ and an integer $n\geq 1$ the configuration space of the spin $O(n)$ model is

$$\Omega := \{\sigma : V(G) \to S^{n-1}\}$$

n is called the number of components

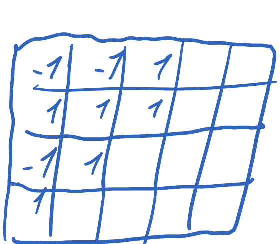
Names: $n=1$: Ising model (then $S^{n-1}=\{-1,1\}$).

$n=2$: XY model (or plane rotator).

$n=3$: Heisenberg model.

The models are also used outside of Physics: E.g., the Ising model can model voting preferences

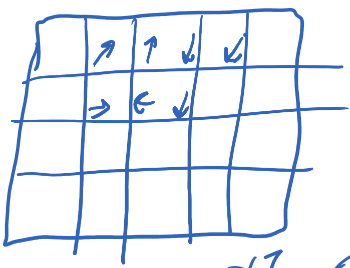
Samples:



$\{-1,1\}$

$n=1$

Ising



$n=2$ $S^1=$ ○

XY

$n=3$

Heisenberg

Energy: The energy, or Hamiltonian, of a configuration $\sigma\in\Omega$ is

$$H(\sigma) = -\sum \langle \sigma_u, \sigma_v \rangle$$

← standard inner product on $\mathbb{R}^n$

$\langle \sigma_u, \sigma_v \rangle = \sum \sigma_{u,i}\sigma_{v,i}$

$$H(\sigma) := -\sum_{u\sim v} \langle \sigma_u, \sigma_v \rangle$$

standard inner product on $\mathbb{R}^n$: $\langle \sigma_u, \sigma_v \rangle = \sum_{k=1}^{n} \sigma_{u,k}\sigma_{v,k}$.

$u \sim v$: i.e. $\{u,v\} \in E(G)$

Temperature parameter: We consider the model at some temperature $T \in (0,\infty]$ ($T=0$ is also possible but not discussed here). Then set $\beta := \frac{1}{T} \in [0,\infty)$ to be the inverse temperature.

Probability measure:

$$d\nu_{G,n,\beta}(\sigma) := \frac{1}{Z_{G,n,\beta}} e^{\overbrace{-\beta H(\sigma)}^{\beta \sum_{u\sim v} \langle \sigma_u, \sigma_v \rangle}} d\sigma$$

Where $d\sigma$ is the uniform dist. on Ω (i.e., product normalized Lebesgue measure on each S^{n-1}).

and $Z_{G,n,\beta} := \int_{\Omega} e^{-\beta H(\sigma)} d\sigma$ is a normalizing factor (the partition function).

Spontaneous magnetization:

With high prob. $\frac{1}{|V(G)|}\sum_{v\in V(G)} \sigma_v$ is bounded away from zero uniformly in the graph size in a certain class of graphs

Choice of graph: We study these models on d-dimensional lattices. For concreteness, we choose $G = \mathbb{T}_L^d$ – the d-dimensional torus of side length $2L$.

$V(G)=\{-L+1,-L+2,\dots,L\}^d$.

There is an edge between $u,v\in V(G)$ if they differ in exactly one coordinate, in which the difference is 1 modulo $2L$.

$d=2,\ L=3$

Results and conjectures

Let $\rho_{x,y}:=\mathbb{E}_{\mathbb{T}^d_L,n,\beta}(\langle\sigma_x,\sigma_y\rangle)$ - the correlation between the spins σ_x,σ_y for $x,y\in V(\mathbb{T}^d_L)$ in the spin $O(n)$ model at inverse temp. β.

We study the magnitude of $\rho_{x,y}$ for far-away $x,y\in V(G)$, uniformly in L, for the various combinations of the number of components n and the spatial dimension d.

Non-negativity and monotonicity:

$\rho_{x,y}\geq 0$ always (on any graph).
(Griffiths)

It is natural to expect $\rho_{x,y}$ to be non-decreasing as β increases.

It is natural to expect $\rho_{x,y}$ to be non-decreasing as β increases.
This is known to occur for $n=1$ (Ising) Griffiths and $n=2$ (XY, by Ginibre).
It is an open question whether there is monotonicity for $n \geq 3$.

High temperature and one dimension:

For each $n \geq 1, d \geq 1$ there exists $\beta_0(n,d)$ s.t. if $\beta < \beta_0(n,d)$ then

$$\rho_{x,y} \leq C_{n,d,\beta} \exp(-c_{n,d,\beta} \underbrace{\|x-y\|_1}_{}).$$

graph dist. of $x,y \in V(\mathbb{T}_L^d)$.

When $d=1$, $\rho_{x,y} \leq C_{n,\beta} \exp(-c_{n,\beta}\|x-y\|_1)$

for all $\beta < \infty$ (and all $x,y \in V(\mathbb{T}_L^d)$).

We thus mostly focus on $d \geq 2$ and the low temperature regime.

Ising model: For each $d \geq 2$ $\exists \beta_c(d)$ s.t.

$\beta < \beta_c(d)$: $\rho_{x,y} \leq C_{d,\beta} \exp(-c_{d,\beta}\|x-y\|_1)$

$\beta > \beta_c(d)$: $\rho_{x,y} \geq c_{d,\beta}$ for all x,y.

(Sharp phase transition) Long-range order.

At criticality: At $\beta = \beta_c(d)$ (much researched today)

$d=2$: $\beta_c(2) = \frac{1}{2}\log(1+\sqrt{2})$ (Onsager 44),

$d=2$: $\beta_c(2)=\frac{1}{2}\log(1+\sqrt{2})$ (Onsager '44),

$$\mathbb{E}^{\mathbb{Z}^2}(\sigma_x\sigma_y)\sim C_1\|x-y\|_2^{-\frac{1}{4}}$$ ← critical exponent

as $\|x-y\|_2\to\infty$. Conformal invariance.

General dimensions $d\geq 3$: $\rho_{x,y}\to 0$ ($d=3$ by Aizenman-Duminil-Copin-Sidoravicius 2015) at the critical point.

$d\geq d_0$: $\mathbb{E}^{\mathbb{Z}^d}(\sigma_x\sigma_y)\sim C_d\|x-y\|_2^{-(d-2)}$ as $\|x-y\|_2\to\infty$ (Sakai, 2007)

Spin $O(n)$ with $n\geq 2$ - Continuous-symmetry models

Two dimensions ($d=2$)

The Mermin-Wagner thm.: No continuous-symmetry breaking

Unlike the Ising model, for each $n\geq 2$, when $d=2$ then $\forall\beta<\infty$,

$$\rho_{x,y}\leq C_{n,\beta}\|x-y\|_1^{-c_{n,\beta}}\quad \text{for all } x,y.$$

$\Rightarrow$ No spontaneous magnetization in 2d.

The Berezinskii-Kosterlitz-Thouless phase transition

When $n=2$ and $d=2$ there is a transition from exp. decay to power-law decay of correlations, predicted by Berezinskii and by Kosterlitz-Thouless (topological phase transition).

The celebrated work ... in the ...

[topologi...] ... the celebrated work
proved in the
of Fröhlich-Spencer (1981)

$n=2, d=2, \beta>\beta_0: \rho_{x,y} \geq c_\beta \|x-y\|_1^{-c_\beta}$.

Polyakov's conjecture: For $n\geq 3, d=2$ it is predicted that there is exp. decay at all temp., but this remains a famous open question on the mathematical level.
(maybe, since the group of rotations of S^{n-1} is non-Abelian for $n\geq 3$).

Dimensions $d\geq 3$

The infra-red bound: Long-range order

For all $n\geq 2$ and all $d\geq 3$,

$$\beta>\beta_0(n,d): \frac{1}{|V(\mathbb{T}_L^d)|^2} \sum_{x,y\in V(\mathbb{T}_L^d)} \rho_{x,y} \geq c_{n,d,\beta}.$$

based on reflection positivity

Much more difficult to prove than for the Ising model and proof technique is fragile — does not apply to related models that should exhibit the same type of behaviour.

We lack techniques for proving continuous-symmetry breaking!

E.g., the above result is expected but unproved

$$H(\sigma) = -\sum(\langle\sigma_u,\sigma_v\rangle)^3.$$

E.g., the ...

when $H(\sigma) = -\sum_{u\sim v} (\langle \sigma_u, \sigma_v \rangle)^3$.

Proofs

High temperature regime - exp. decay of correlations

(General approach by Dobrushin - Dobrushin uniqueness theorem)

We will show it with an approach more tailored to the specific models.

High temperature expansion: Fix a finite graph G.

$$Z_{G,n,\beta} = \int_\Omega \prod_{u\sim v} e^{\beta\langle \sigma_u, \sigma_v \rangle} d\sigma =$$

$$= \int_\Omega \prod_{u\sim v} \Big(e^{-\beta} + \underbrace{\big(e^{\beta\langle \sigma_u, \sigma_v \rangle} - e^{-\beta}\big)}_{\text{This is close to 0 for } \beta \text{ small, and it is non-negative}}\Big) d\sigma =$$

$$= e^{-\beta|E(G)|} \int_\Omega \prod_{u\sim v} \Big(1 + \underbrace{\big(e^{\beta(\langle \sigma_u, \sigma_v \rangle + 1)} - 1\big)}_{f_\beta(\sigma_u, \sigma_v)}\Big) d\sigma =$$

$$= e^{-\beta|E(G)|} \cdot \sum_{E \subseteq E(G)} \underbrace{\int_\Omega \prod_{\{u,v\}\in E} f_\beta(\sigma_u, \sigma_v)\, d\sigma}_{Z(E)}.$$

subgraph

probabilistic interpretation of the expansion:

Probabilistic interpretation of the expansion:

We may sample σ from $\mu_{G,n,\beta}$ using a two-step procedure:

1) Sample a random subgraph $E \subseteq E(G)$ with prob. that $E = E_0$ proportional to $Z(E_0)$. $\left(\mathbb{P}(E=E_0) = \frac{Z(E_0)}{\sum_{E_1 \subseteq E(G)} Z(E_1)}\right)$.

2) Given E, we sample σ with prob. density (wrt. $d\sigma$) proportional to

$$Z(E,\sigma) = \prod_{\{u,v\} \in E} f_\beta(\sigma_u, \sigma_v).$$

Then $\sigma \sim \mu_{G,n,\beta}$.

In other words we have a joint dist. on (E,σ) whose second marginal is dist. $\mu_{G,n,\beta}$.

$n=1$: The joint dist. is called the Edwards-Sokal coupling, and the marginal on E is called the random-cluster model and has the explicit formula:

$$\mathbb{P}(E=E_0) = 2^{K(E_0)} p^{|E_0|} (1-p)^{|E(G) \setminus E_0|}$$

with $p = 1 - e^{-2\beta}$ and $K(E_0)$ which is the number of connected comp. in E_0 viewed on the vertex set $V(G)$.

As a result of the two-step sampling procedure we may write

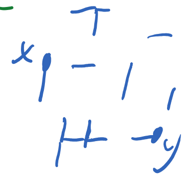
The random edge set

As a result …
We may write

The random edge set

$$\rho_{x,y} = \mathbb{E}(\langle \sigma_x, \sigma_y \rangle) = \mathbb{E}(\mathbb{E}(\langle \sigma_x, \sigma_y \rangle / E)) =$$

$$= \mathbb{E}(\mathbb{E}(\langle \sigma_x, \sigma_y \rangle \underbrace{\mathbb{1}_{x \overset{E}{\leftrightarrow} y}}_{x \text{ is connected to } y \text{ in } E} / E)) +$$



$$+ \mathbb{E}(\mathbb{E}(\langle \sigma_x, \sigma_y \rangle \mathbb{1}_{x \overset{E}{\not\leftrightarrow} y} / E))$$

Cond. on E, when $x \overset{E}{\not\leftrightarrow} y$ then σ_x, σ_y are indep. and uniform on S^{n-1}. Thus the second term is zero.

We also use that $|\langle \sigma_x, \sigma_y \rangle| \le 1$, to get

$$\underline{\le} \; \mathbb{E}(\mathbb{E}(\mathbb{1}_{x \overset{E}{\leftrightarrow} y} / E)) = \mathbb{P}(x \overset{E}{\leftrightarrow} y).$$

equality when $n=1$

We prove that E is very sparse, typically. Specifically, we first show that

$\forall e \in E(G)$, $E_0 \subseteq E(G) \setminus \{e\}$,

$$\mathbb{P}(e \in E \,/\, E \setminus \{e\} = E_0) \le 1 - e^{-2\beta} \quad (*).$$

Indeed,

$$\mathbb{P}(e \in E / E \setminus \{e\} = E_0) = \frac{\mathbb{P}(E = E_0 \cup \{e\})}{\mathbb{P}(E \setminus \{e\} = E_0)} =$$

$$= \frac{\mathbb{P}(E = E_0 \cup \{e\})}{\mathbb{P}(E = E_0 \cup \{e\}) + \mathbb{P}(E = E_0)} =$$

$$= \frac{1}{1 + \frac{\mathbb{P}(E = E_0)}{\mathbb{P}(E = E_0 \cup \{e\})}}$$

only involves ratios of $Z(E_1)$.

$$= \frac{1}{1 + \frac{Z(E_0)}{Z(E_0 \cup \{e\})}}.$$

Then $$\frac{Z(E_0)}{Z(E_0 \cup \{e\})} = \frac{\int_{\Omega} \prod_{\{u,v\} \in E_0} f_\beta(\sigma_u, \sigma_v)\, d\sigma}{\int_{\Omega} \prod_{\{u,v\} \in E_0 \cup \{e\}} f_\beta(\sigma_u, \sigma_v)\, d\sigma} \geq$$

$\forall \sigma_u, \sigma_v : f_\beta(\sigma_u, \sigma_v) \leq e^{2\beta} - 1$

$$\geq \frac{1}{e^{2\beta} - 1}$$ and this proves (*).

Consequently, $\forall e_1, \ldots, e_K \in E(G)$ distinct,

$$\mathbb{P}(e_1, \ldots, e_K \in \mathbb{E}) \leq (1 - e^{-2\beta})^K.$$

For a graph G with maximal degree Δ_G, when $\beta < \beta_0(\Delta_G)$, this implies that $\mathbb{P}(x \overset{E}{\longleftrightarrow} y) \leq C_G \exp(-c_G d_G(x,y))$

graph dist. in G between x and y

by a union bound over all paths between x and y.

For $G = \mathbb{T}_L^d$ get exp. decay for
$$\beta < \frac{1}{2} \log\left(\frac{2d-1}{2d-2}\right).$$

In particular, always when $d = 1$.